

Spin glass behavior upon diluting frustrated magnets and spin liquids: a Bethe-Peierls treatment^{*}

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Abstract. A Bethe-Peierls treatment to dilution in frustrated magnets and spin liquids is given. A spin glass phase is present at low temperatures and close to the percolation point as soon as frustration takes a finite value in the dilute magnet model; the spin glass phase is reentrant inside the ferromagnetic phase. An extension of the model is given, in which the spin glass/ferromagnet phase boundary is shown not to reenter inside the ferromagnetic phase asymptotically close to the tricritical point whereas it has a turning point at lower temperatures. We conjecture similar phase diagrams to exist in finite dimensional models not constraint by a Nishimori's line. We increase frustration to study the effect of dilution in a spin liquid state. This provides a “minimal” ordering by disorder from an Ising paramagnet to an Ising spin glass.

PACS. 61.43.-j Disordered solids – 64.60.Ak Renormalization-group, fractal, and percolation studies of phase transitions – 64.60.Kw Multicritical points

1 Introduction

When non magnetic sites are diluted in an unfrustrated ferromagnet with a probability μ , the transition temperature is reduced and vanishes at the percolation threshold $1 - \mu_{P,0}$. In such non frustrated systems, two phases only exist: a low temperature ferromagnetic phase above the percolation threshold, and a paramagnetic phase. If the temperature is decreased at the percolation threshold, the dynamics becomes slower because large-scale droplet-like objects of size ξ_T form, with $\ln \xi_T \sim J/T$ [1]. These objects have energy barriers scaling like the logarithm of their volume [2,3]. This results in a slow dynamics and interrupted aging [4] (*i.e.* with a finite relaxation time [5]). This shows that despite the absence of frustration, the simplest models of dilute magnets already have a phenomenology close to the one of spin glasses, even though freezing in these systems is a cross-over due to an increasing correlation length becoming of order of the system size. This indicates that some perturbations of these unfrustrated systems may drive them to a true spin glass phase, which we show in the present article by studying the thermodynamics of a particular model.

In dilute magnet compounds, such as $\text{Eu}_x\text{Sr}_{1-x}\text{S}$ [6–8], a low temperature spin glass phase appears close to the percolation threshold. The main features of the phase diagram are:

- (i) As the dilution μ is increased from the pure system with $\mu = 0$, the ordering temperature decreases.
- (ii) A tricritical point exists at a dilution μ_t and temperature T_t , with $1 - \mu_t$ of order of the percolation threshold $1 - \mu_{P,0}$ in the absence of frustration. At this tricritical point, the ferromagnetic, paramagnetic and spin glass phases meet.
- (iii) As dilution is increased from μ_t , the spin glass transition temperature decreases from T_t at the tricritical point to zero at the percolation threshold $1 - \mu_P$, with $1 - \mu_P$ the percolation threshold of the system with frustration, smaller than the percolation threshold $1 - \mu_{P,0}$ in the absence of frustration.
- (iv) The spin glass phase is reentrant inside the ferromagnetic phase.

One purpose of the present article is to show that these qualitative features of the phase diagram can be reproduced in a model that combines dilution and short range frustration. This model does not consist in a detailed microscopic modeling of $\text{Eu}_x\text{Sr}_{1-x}\text{S}$, but rather contains the generic ingredients entering the physics of these systems (dilution and short range frustration) [9]. Even though this treatment relies on a specific lattice topology (a tree structure), Bethe-Peierls phase diagrams are equivalent to mean field phase diagrams while the Bethe-Peierls method is powerful enough to give an exact answer to the issue of

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reentrance. The resulting phase diagrams are therefore not expected to be specific to our treatment but are generic features of the coupling Hamiltonian.

The article is organized as follows. The model is given in Section 2. The paramagnetic phase boundary is solved in Section 3. We study in Section 4 the spin glass/ferromagnet phase boundary and show that it is reentrant. The issue of reentrance is non trivial because Nishimori’s argument [10–14] does not hold in our model, part of the exchanges being frozen. An extension of the model is given in which the spin glass/ferromagnet phase boundary is shown not to reenter in the ferromagnetic phase asymptotically close to the tricritical point, whereas it has a turning point at a lower temperature. We expect this unusual type of phase diagram to be a generic feature of models combining disordered and frozen exchanges, and may be obtained in finite dimensional models also. Finally, we study in Section 5 the effect of diluting a spin liquid state, in which case ordering by disorder generates a transition from an Ising paramagnet to an Ising spin glass. This ordering by disorder mechanism is “minimal” because spins have an Ising symmetry only.

2 The model

2.1 The dilute magnet model and its generalization

In a Bethe-Peierls calculation, only the properties of the “top” spin (the highest one in the hierarchy; see Fig. 1) are considered, and the thermodynamic limit is obtained by growing the number of generations to infinity [15–17]. The top spin fixed point magnetization distributions $P^*(m)$ are of three types: (i) *paramagnetic phase*: $P^*(m) = \delta(m)$; (ii) *spin glass phase*: $P^*(m)$ is even; (iii) *ferromagnetic phase*: $P^*(m)$ has a finite first moment. The transitions between the phases are of a mean field type [15–17]. The phase diagram of the $\pm J$ model on the Bethe lattice [18–24] is very similar to the one of the Sherrington-Kirkpatrick model, and the Bethe-Peierls treatment allows a correct description [24] of the Almeida-Thouless line [25]. This shows that a Bethe-Peierls treatment succeeds in reproducing the mean field phase diagram of spin glass models.

We consider a model in which the ferromagnetic bonds J of the Cayley tree are canceled with a probability μ . Frustration is added by completing the triangles, with frozen antiferromagnetic bonds τ forming a Husimi cactus-like structure [26] (see Fig. 1). We use the binary variables $\theta_i = 0, 1$, with $J_i = \theta_i J$. The temperature is expressed in units of J . The Hamiltonian is

$$H = - \sum_{\langle i,j \rangle} \theta_{i,j} \sigma_i \sigma_j + \tau \sum_{\langle i,j \rangle'} \sigma_i \sigma_j, \quad (1)$$

where $\langle i,j \rangle$ denotes the bonds of the tree structure and $\langle i,j \rangle'$ the next nearest neighbor pairs of sites in the same generation. The distribution of the θ -variable is

$$p(\theta) = (1 - \mu)\delta(\theta - 1) + \mu\delta(\theta). \quad (2)$$

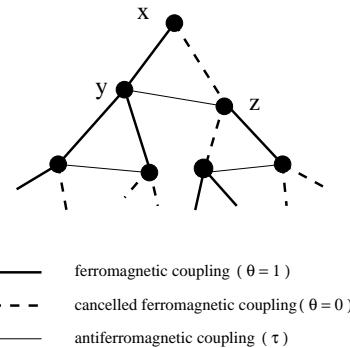


Fig. 1. The Husimi cactus-like structure model of dilute magnet with frustration. The structure with a top spin x is obtained by gluing the two structures with top spins y and z . The top spin is the highest one in the hierarchy (at site x in this figure). The ferromagnetic bonds of the tree are canceled with a probability μ and a fixed antiferromagnetic coupling τ is added.

The specificity that some bonds are frozen in this model has drastic consequences on the shape of the phase diagram, as we will show.

The Husimi cactus structure allows the introduction of a local frustration resulting from next-nearest-neighbor interactions, and can be used to mimic the effects of local antiferromagnetic interactions in dilute compounds. Chandra and Douçot [27] considered a frustrated spin model on a regular Husimi cactus structure, and studied ordering by disorder in the spin liquid state in the Bethe-Peierls limit (see also [28] for a study of the effect of quantum fluctuations). These authors considered a non disordered model in which a spin glass phase cannot exist [27]. In our model with randomness, we show the stability of a spin glass solution in some regions of the phase diagram.

We consider bond instead of site percolation because the site percolation threshold of the Husimi cactus structure would be equal to the one of the tree structure, independent of the additional bonds τ . The bond percolation model is therefore better suited for modeling dilute compounds [6], since the bond percolation threshold of the structure without frustration (a tree structure) is $1 - \mu_{P,0} = 1/2$, larger than the bond percolation threshold $1 - \mu_P = 1 - 1/\sqrt{2}$ of the structure with frustration (the Husimi cactus structure shown in Fig. 1).

For the sake of generality, we not only consider a dilute magnet model with $\theta = 0, 1$, but extend the bond distribution (2) to incorporate possible antiferromagnetic bonds on the tree structure. The distribution of the bond variables θ is

$$p(\theta) = (1 - \lambda)(1 - \mu)\delta(\theta - 1) + \mu\delta(\theta) + \lambda(1 - \mu)\delta(\theta + 1), \quad (3)$$

while the additional antiferromagnetic bonds τ are frozen. This model interpolates between the dilute model with a short-range frustration τ ($\lambda = 0$), and the $\pm J$ model ($\mu = 0$ and $\tau = 0$).

2.2 Absence of a Nishimori line argument

Our Hamiltonian is formally invariant under local gauge transformations $\sigma_i \rightarrow \epsilon_i \sigma_i$, $J_{i,j} \rightarrow \epsilon_i \epsilon_j \sigma_i \sigma_j$, with $\epsilon_i = \pm 1$ [29]. In some spin glass models (such as the $\pm J$ model), gauge invariance provides strong constraints on the phase diagram. The internal energy can be calculated exactly on Nishimori's line by expanding the average energy over the gauge group [10]. This line crosses the phase boundary at the tricritical point [10–13]. Moreover, spin correlations can be related to gauge variable correlations, with the consequence that the frontier between the ferromagnetic and spin glass phases is either vertical or reentrant [10]. In our model, gauge invariance is useless for the following reason. One can define a local distribution of bond variables $P_{i,j}(J)$, being (3) on the tree bonds, and $\delta(J + \tau)$ on the antiferromagnetic bonds τ . Nishimori's line is defined by $\beta_N = \frac{1}{2} \ln(\frac{1-\lambda}{\lambda})$ [14], and $\beta_N = +\infty$. The first equality originates from the tree bond variables and the second from the frozen antiferromagnetic bonds τ . The two equalities can be formally met if $\lambda = 0$, in which case Nishimori line is $\beta_N = +\infty$. However, this does not make predictions on the phase diagram possible even in the case $\lambda = 0$ [30] because one does not expect to be able to describe finite temperature spin glass properties in terms of the ground state only (the only state selected if $\beta_N = +\infty$). Nishimori's argument can therefore not be made in this model, and the question of reentrance (item (iv) in the introductory section) cannot be answered on the basis of Nishimori's line while Bethe-Peierls calculations are powerful enough to allow the derivation of exact results. We show that the spin glass/paramagnet boundary is reentrant in the dilution model ($\lambda = 0$) in the (μ, T) plane. In the $\pm J$ model with the additional coupling τ ($\mu = 0$), and in the (λ, T) plane, we show that the spin glass/ferromagnet phase boundary is not reentrant asymptotically close to the tricritical point, whereas it has a turning point at lower temperatures. This behavior, richer than in usual spin glass models, is to our opinion a generic feature of Hamiltonians combining disorder and frozen bonds. We conjecture the existence of finite dimensional models with a similar phase diagram.

3 Recursion relations

We now derive the recursion of the top-site magnetization when cacti are glued as shown in Figure 1. We denote by m_x the magnetization at site x , and m_y and m_z the magnetizations at the descendant sites y and z . The derivation of the recursion relations in our dilute magnet model is similar to the case of the $\pm J$ model (see Ref. [23]).

3.1 Recursions and the paramagnet phase boundary

Following reference [23], we denote by $\mathcal{Z}_x^{(\pm)}$ the conditional partition function with the spin at site x frozen in the direction \pm . The magnetization at site x is $m_x =$

$(\mathcal{Z}_x^{(+)} - \mathcal{Z}_x^{(-)}) / (\mathcal{Z}_x^{(+)} + \mathcal{Z}_x^{(-)})$. The partition functions $\mathcal{Z}_x^{(\pm)}$ are related to the partition functions $\mathcal{Z}_{y,z}^{(\pm)}$ of the descendant sites according to

$$\mathcal{Z}_x^{(\sigma_x)} = \sum_{\sigma_y, \sigma_z} W_{\theta_y, \theta_z}^B(\sigma_x | \sigma_y, \sigma_z) \mathcal{Z}_y^{(\sigma_y)} \mathcal{Z}_z^{(\sigma_z)}, \quad (4)$$

with the Boltzmann weight factor

$$W_{\theta_y, \theta_z}^B(\sigma_x | \sigma_y, \sigma_z) = \exp(\beta(\theta_y \sigma_x \sigma_y + \theta_z \sigma_x \sigma_z)) \times \exp(-\beta \tau \sigma_y \sigma_z), \quad (5)$$

and $\theta = 0, \pm 1$. We next trace over the spins at sites y and z in equation (4) to obtain

$$m_x = f(m_y, m_z | \theta_y, \theta_z) = p \frac{m_y(\theta_y - u\theta_z) + m_z(\theta_z - u\theta_y)}{1 - up^2\theta_y\theta_z + m_y m_z(p^2\theta_y\theta_z - u)}, \quad (6)$$

with $p = \tanh(\beta J)$ and $u = \tanh(\beta \tau)$. The recursion of the magnetization distribution is

$$P_{n+1}(m_x) = \int dm_x dm_y \sum_{\theta_y, \theta_z} p(\theta_y) p(\theta_z) P_n(m_y) \times P_n(m_z) \delta(m_x - f(m_y, m_z | \theta_y, \theta_z)), \quad (7)$$

with $p(\theta)$ the distribution of bond variables (3), and P_n the magnetization distribution of the top spin with n levels of hierarchy. We denote by $\langle\langle m^k \rangle\rangle_n$ the moment of order k of $P_n(m)$.

We now parametrize the tricritical line, where the three phases (paramagnetic, ferromagnetic and spin glass) meet. The meeting point of these phases is a line in the parameter space (λ, μ, T) . If λ [μ] is fixed and the phase diagram is considered in the (μ, T) [(λ, T)] plane, the three phases meet in a tricritical point. Let us first consider the stability of the paramagnetic solution with respect to perturbations in the first moment. To lowest order the recursion of the first moment is $\langle\langle m \rangle\rangle_{n+1} = 2p \langle\langle G_{y,z} \rangle\rangle \langle\langle m \rangle\rangle_n$, with $G_{y,z} = (\theta_y - u\theta_z) / (1 - up^2\theta_y\theta_z)$. The disorder average of G is understood as $\langle\langle G_{y,z} \rangle\rangle = \sum_{\theta_y, \theta_z} p(\theta_y) p(\theta_z) G_{y,z}$. The paramagnetic solution is stable with respect to perturbations in the first moment if $2p \langle\langle G_{y,z} \rangle\rangle < 1$. A similar reasoning shows that the paramagnetic solution is stable with respect to perturbations in the second moment if $2p^2 \langle\langle G_{y,z}^2 \rangle\rangle < 1$. To summarize, the tricritical line is defined by

$$2p \langle\langle G_{y,z} \rangle\rangle = 1, \text{ and } 2p^2 \langle\langle G_{y,z}^2 \rangle\rangle = 1. \quad (8)$$

3.2 Limiting cases

3.2.1 The pure system

Let us consider the recursion (6) in the pure system limit in which the variables θ are all equal to unity. This amounts to specializing the distribution (3) to the

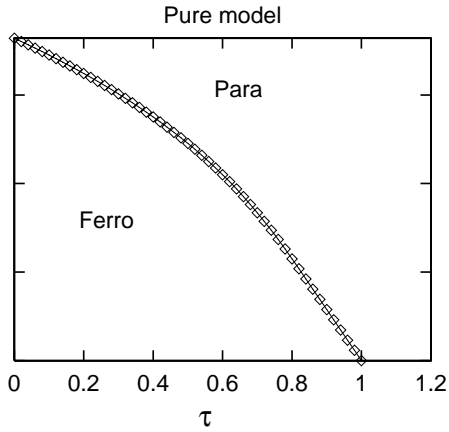


Fig. 2. Phase diagram of the pure Husimi cactus system (all the coupling θ being unity) as a function of the frustration τ . A low-temperature ferromagnetic phase is present if $\tau < 1$. If $\tau > 1$, the system is a spin liquid (it does not order even at zero temperature).

case $\lambda = \mu = 1$ while keeping finite the local frustration τ . The only possible phases are ferromagnetic and paramagnetic. The recursion of the magnetization is $m_{n+1} = 2p(1-u)m_n / (1-pu^2 + (p^2-u)m_n^2)$. The paramagnetic phase is stable against ferromagnetic fluctuations if $2p(1-u)/(1-pu^2) < 1$. The phase diagram is shown in Figure 2 as a function of the frustration τ with a spin liquid phase if $\tau > 1$. When considering in the following the frustrated magnet model, we assume $\tau < 1$, in which case the pure system has an ordered phase at low temperature. Diluting the spin liquid state with $\tau > 1$ is examined in Section 5.

3.2.2 $\pm J$ model

The $\pm J$ model is recovered if $\mu = \tau = 0$, with a tricritical point at coordinates $p_t = 1/2$, $\lambda_t = \frac{1}{2} \left(1 - \frac{1}{\sqrt{2}}\right)$ [23].

3.2.3 Dilute magnet with frustration

If $\lambda = 0$ and $\tau \neq 0$ the relations (8) become

$$\frac{2p(1-\mu)(1-u)(1-\mu up^2)}{1-up^2} = 1 \quad (9)$$

$$\frac{2(1-\mu)p^2}{(1-up^2)^2} ((1-\mu)(1-u)^2 + \mu(1+u^2)(1-up^2)^2) = 1, \quad (10)$$

determining the paramagnetic phase boundary shown in Figure 3. The frontier between the spin glass and ferromagnetic phases will be examined in Section 4 by looking for an instability in the first moment of the spin glass solution.

The zero temperature limit of the paramagnetic/spin glass phase boundary can be obtained by considering first

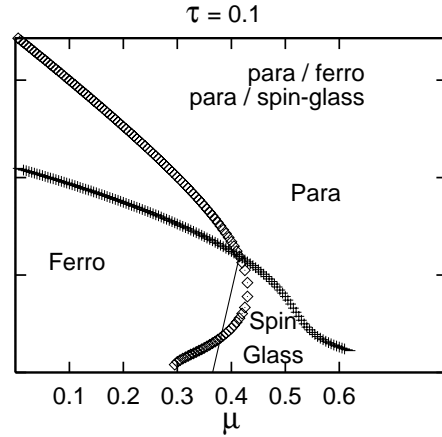


Fig. 3. Phase diagram of the dilute magnet model with frustration ($\lambda = 0$, $\tau = 0.1$). The spin glass phase exists below the percolating dilution $\mu_P = 1/\sqrt{2} \simeq 0.707$. The paramagnetic/spin glass phase boundary inside the ferromagnetic phase is unphysical, as well as the paramagnetic/spin glass boundary inside the spin glass phase. The solid line is obtained from the calculation in Section 4.3 of the frontier between the spin glass and ferromagnetic phases. This solution is exact close to the critical point and we have continued it to lower temperatures by an arbitrary linear behavior. The exact zero temperature spin glass/ferromagnet phase boundary is $\mu_0 \simeq 0.24191$.

the limit $p = 1$ in equation (10), and second the limit $u = 1$. The order in which the two limits are taken is imposed by the fact that $1-p \ll 1-u \ll 1$ at low temperatures because $\tau < 1$. One finds the limit to be $1-\mu = 1-1/\sqrt{2}$. As it is expected, this dilution is equal to the percolation threshold of the Husimi cactus structure.

4 Frontier between the spin glass and ferromagnetic phases

4.1 Method

In order to determine the frontier between the spin glass and ferromagnetic phases, we study the instability of the spin glass solution with respect to perturbations in the first moment. This involves first calculating the spin glass solution close to the tricritical line and next determining whether this solution is stable with respect to ferromagnetic fluctuations. Carlson *et al.* [23] performed this calculation for the $\pm J$ model close to the tricritical point, and shown the spin glass phase to be marginally reentrant inside the ferromagnetic phase. By marginal, we mean that the spin glass/ferromagnet phase boundary has a quadratic behavior $\lambda_t - \lambda \sim (T_t - T)^2$, which is specific to this model. Other models (as the one we presently analyze) have a linear behavior $\lambda_t - \lambda \sim T_t - T$, with a positive (reentrant behavior) or negative prefactor (non reentrant behavior). The calculation follows reference [23] where the $\pm J$ model was solved, and is asymptotically exact close to the tricritical point. This is complemented by an exact

determination of the zero temperature phase boundaries which, to our knowledge, has not appeared previously in the literature even for the $\pm J$ model.

We first determine the asymptotic spin glass solution close to the tricritical line. The second moment to lowest order is

$$\langle\langle m^2 \rangle\rangle = \frac{2p^2 \langle\langle G_{y,z}^2 \rangle\rangle - 1}{4p_t^2 \langle\langle G_{y,z} G_{z,y} H \rangle\rangle_t}, \quad (11)$$

with $H = (p^2 \theta_y \theta_z - u)/(1 - up^2 \theta_y \theta_z)$, and the subscript “t” denoting a quantity evaluated on the tricritical line. We next consider a perturbation in the first moment of the spin glass solution. The recursion of the first moment is $\langle\langle m \rangle\rangle_{n+1} = \kappa \langle\langle m \rangle\rangle_n$, with $\kappa = 2p \langle\langle G_{y,z} \rangle\rangle - 2p_t \langle\langle m^2 \rangle\rangle \langle\langle G_{y,z} H \rangle\rangle_t$ to order $(T_t - T)$. If $\kappa < 1$ the spin glass phase is stable and otherwise the ferromagnetic phase is stable.

4.2 $\pm J$ model with an additional short-range coupling τ ; Fixed μ ; (λ, T) phase diagram

In the small- τ limit, it is straightforward to show that

1. The tricritical point can be determined in an expansion in u_t : $p_t = 1 - 2\lambda_t = \frac{1}{\sqrt{2}}(1 + \frac{u_t}{4})$.
2. The slope of the spin glass/ferromagnet phase boundary at the tricritical point is

$$\left. \frac{d\kappa}{dT} \right|_t = \frac{u_t}{4T_t^0} \left(1 - \frac{5\sqrt{2}}{2T_t^0} \right) \simeq -0.467 u_t < 0, \quad (12)$$

with $T_t^0 = 1/\tanh^{-1}(1/\sqrt{2}) \simeq 1.135$ the tricritical point temperature with $\tau = 0$.

From what we deduce that the spin glass phase does not reenter inside the ferromagnetic phase close to the tricritical point.

Notice that $d\kappa/dT|_t$ in equation (12) vanishes if $\tau = 0$. This is because reentrance is marginal in the $\pm J$ Bethe lattice spin glass [23] and can therefore not be obtained from an expansion to first order in $T_t - T$. We have evaluated numerically the coefficient κ with a finite τ and a finite μ . As μ is increased above a critical value, the transition changes from reentrant to non reentrant.

We now derive the exact spin glass solution in the zero temperature limit, which allows to discriminate rigorously between the two behaviors in Figure 4a and 4b. We look for the zero temperature fixed point spin glass and ferromagnetic solutions $P^*(m)$ under the form

$$P^*(m) = \frac{x+y}{2} \delta(m-1) + (1-x) \delta(m) + \frac{x-y}{2} \delta(m+1), \quad (13)$$

with x and y the spin glass and ferromagnet order parameters. It turns out that the functional form of the magnetization distribution (13) is stable when it is iterated in the zero temperature limit of (6) and (7). To determine x and y , we impose (13) to be the fixed point magnetization

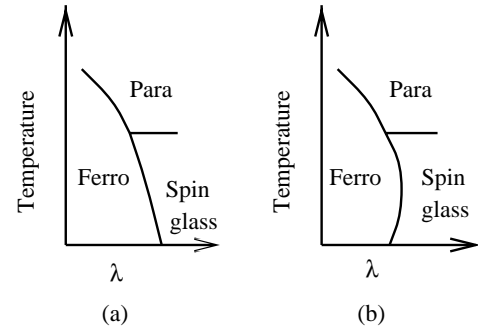


Fig. 4. Possible shapes of the spin glass/ferromagnet phase boundary in the $\pm J$ model with a small antiferromagnetic coupling τ corresponding to a spin glass phase not reentrant inside the ferromagnetic phase asymptotically close to the tricritical point. This implies two possible behaviors: (a): no reentrance at any temperature (b): no reentrance close to the tricritical point but reentrance at lower temperatures. We prove that (b) is the correct behavior in a zero temperature exact solution.

distribution. The solution with a finite magnetization is $x = (1 - 4\lambda)(1 - 2\lambda)$, and $y^2 = (1 - 4\lambda)(1 - 8\lambda)/(1 - 2\lambda)^2$. Imposing $y^2 > 0$ leads to the intersection $\lambda_0 = 1/8 = 0.125$ of the spin glass/ferromagnet phase boundary and the zero temperature axis in the (λ, T) plane. λ_0 is independent of the strength of the additional coupling τ . If $\tau = 0$, the value λ_t of λ at the tricritical point is $\lambda_t = \frac{1}{2}(1 - \frac{1}{\sqrt{2}}) \simeq 0.146$ [23], larger than λ_0 . The zero temperature solution is therefore consistent with the reentrant behavior of the spin glass/ferromagnet phase boundary of the $\pm J$ Bethe lattice spin glass [23]. As τ is increased, the tricritical point $(\lambda_t(\tau), T_t(\tau))$ evolves with τ whereas the intersection of the spin glass/ferromagnet phase boundary and the zero temperature axis remains equal to λ_0 . Therefore, if τ is small, $\lambda_t(\tau)$ remains larger than λ_0 . From what we deduce the existence of a turning point in the spin glass/ferromagnet phase boundary (Fig. 4b): the spin glass phase does not reenter close to the tricritical point whereas it reenters at lower temperatures. We believe this behavior to be generic of spin glass models with frozen exchanges and we conjecture that a similar behavior may be obtained in finite dimensional models.

4.3 Dilution with a short-range frustration τ ; $\lambda = 0$; (μ, T) phase diagram

A small- τ perturbation calculation leads to $p_t = 1 - \frac{1}{2}u_t$, $\mu_t = \frac{1}{2} - \frac{1}{2}u_t$, and $d\kappa/dT|_t = \tau/2T_t^2 > 0$, which proves that the spin glass phase reenters in the ferromagnetic phase close to the tricritical point in the limit of small τ . We have shown in Figure 3 the behavior of the spin glass/ferromagnet phase boundary. This phase boundary is exact only close to the tricritical point, and we have continued it by an arbitrary straight line at lower temperatures. The reentrant behavior is confirmed by zero temperature exact results. The paramagnetic/spin glass

frontier intersects the zero temperature axis at the percolation threshold, and the spin glass/ferromagnet frontier intersects the zero temperature axis at μ_0 , the real root of $-10\mu_0^3 + 6\mu_0^2 - 5\mu_0 + 1 = 0$, approximately $\mu_0 \simeq 0.24191$. This confirms the reentrant behavior of the spin glass transition in the dilute magnet model with frustration.

5 Diluting the spin liquid

We now consider dilution in the regime $\tau > 1$, *i.e.* when the pure system is a spin liquid (see Fig. 2). As it could be expected, a ferromagnetic instability of the spin liquid solution (Eq. (9)) does not exist. However, a spin glass instability of the paramagnetic solution does exist upon diluting the system. Let us first consider the zero temperature phases, in the limit $1 - u \ll 1 - p \ll 1$ (since $\tau > 1$). The phase diagram at finite temperatures is shown in Figure 5. A finite temperature spin glass phase opens from the point ($\mu = 1/2$, $T = 0$) as temperature is increased from zero. The low temperature phase boundary is $T = 4(1 - \tau)/[\ln(\mu - 1/2)^2]$. This provides a simple situation in which diluting an Ising spin liquid results in an Ising spin glass phase. The underlying ordering by disorder mechanism [36] is analyzed in Section 6.2.

6 Conclusion: diluting a frustrated magnet versus diluting a spin liquid

We have shown the existence of a spin glass solution upon diluting both the weakly frustrated magnet ($\tau < 1$), and the spin liquid ($\tau > 1$). We underline the differences in the physics in these two regimes.

6.1 Diluting the frustrated ferromagnet

We believe the generation of a spin glass phase upon weakly frustrating a dilute magnet close to the percolation threshold to be due to the following: the strong diluted unfrustrated magnet is already close to a spin glass. This can be seen on the example of square lattice dilute magnets [2], where dilution removes sites in the ferromagnet up to the point where the percolating cluster is a fractal object at the percolation point. Since the order of ramification of percolating clusters is finite [32], one can isolate large droplet-like objects [33] from the remaining of the structure by cutting a finite number of bonds. This results in large sets of spins that can be reversed at a finite energy cost, thus being responsible for the existence of quasi-degenerate ground states separated by a large distance in phase space (with different magnetizations [34]), and with barriers scaling like the logarithm of their volume [2].

The addition of frustration in dilute magnets close to the percolation threshold turns the quasi spin glass order into a true one. We have shown this explicitly in our model, and a similar behavior was obtained in another

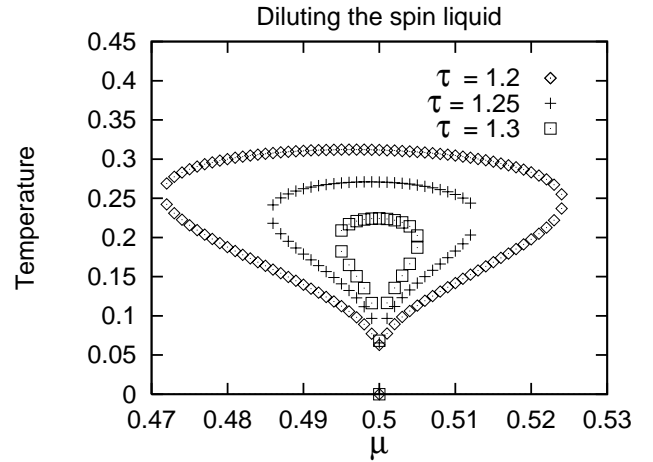


Fig. 5. Boundary between the spin glass and paramagnetic phases upon diluting the spin liquid state of our model ($\tau > 1$). The spin glass phase is confined inside the boundary shown for $\tau = 1.2$ (\diamond), $\tau = 1.25$ ($+$), and $\tau = 1.3$ (\square). The spin glass boundary collapses onto the point $\mu = 1/2$ in the zero temperature limit. This boundary behaves like $T \sim 1/\ln|\mu - 1/2|$ around this singular point. The spin glass phase is favored upon increasing the temperature (ordering by disorder – see Sect. 6.2).

model [12]. We do not expect the low energy states of the spin glass phase with a small frustration to be very different from the droplet-like states of the unfrustrated magnet.

The “chaos and memory” behavior of metallic spin glasses was put forward in reference [35], associated to the growth of fractal droplets with a chaotic behavior in the sense that droplets at a given temperature overlap weakly with droplets at a different temperature. We do not expect a chaotic behavior in our model because the finite temperature droplet excitations of the frustrated dilute magnet should be obtained from reversing clusters of spins in the unfrustrated magnet dilute lattice.

6.2 Diluting the spin liquid: ordering by disorder

The mechanism for generating a spin glass phase from the spin liquid is different. The spin glass phase originates from a balance between the small-dilution regime in which dilution suppresses the liquid behavior in favor of spin-glass correlations, and a large-dilution regime in which dilution suppresses spin glass correlations by cutting the system into finite pieces. This is an order by disorder mechanism [36]: thermal fluctuations favor a spin glass arrangement and therefore reduce the phase space dimensionality compared to the one of the spin liquid state. Let us think in terms of low temperature properties in the large- τ limit. In this limit, the neighboring spins coupled by the strong antiferromagnetic exchange τ correlate antiferromagnetic, thus leaving mainly two residual degrees of freedom per bond τ . We note m_y and m_z the magnetization of these two spins, and, for the sake of a qualitative

argument, assume $m_y = -m_z$ as a result of the strong bond τ . Let us assume the spins at sites y and z to be frozen and look whether freezing is relevant in the Bethe-Peierls limit. We see from equation (6) that $m_x = 0$ if the two ferromagnetic bonds are present (*i.e.* if the triangular plaquette is frustrated, $\theta_y = \theta_z = 1$). In the unfrustrated plaquettes $\theta_y = 0, \theta_z = 1$, or $\theta_y = 1, \theta_z = 0$, correlations in the magnetization can propagate from one generation to the other. The system is cut into two pieces if $\theta_y = \theta_z = 0$, preventing correlations to propagate from one generation to the other. When the ferromagnetic bonds are diluted, frustration is reduced since the fraction of frustrated triangular plaquettes $(1 - \mu)^2$ decreases upon increasing the dilution μ . Decreasing frustration therefore decreases the short range liquid-like correlations and favors a cooperative spin glass arrangement. In the large dilution limit, the exchanges are severely depleted and a paramagnetic behavior is restored since the system is cut into finite pieces. In between these two regimes, the unfrustrated bond configurations $\theta_y = 1, \theta_z = 0$ and $\theta_y = 0, \theta_z = 1$ with a weight $2\mu(1 - \mu)$ dominate the physics and make a spin glass order possible.

6.3 Concluding remarks

Finally, we would like to compare the present work to other approaches developed previously in the literature, and mention some open questions. The phase diagram with all the bonds drawn from the distribution (3) was studied by Aharony [37], Giri and Stephen [38] and Viana and Bray [39]. These models share similarities with the dilute fcc antiferromagnets studied by de Seze [40] and Wengel, Henley and Zippelius [41].

Nieuwenhuizen [42] and Nieuwenhuizen and van Duin [43] studied the field theory of a model of site-disordered magnet. One may also define a model similar to ours in a finite dimension. As we conjectured, a phase diagram similar to the one in Figure 4b may be obtained. On the other hand, it may be useful to investigate replica symmetry breaking in Bethe-Peierls calculations.

Hierarchical lattices have been used previously by Georges and Le Doussal [12] in relation with the renormalization group flow along Nishimori's line, and by Gingras and Sørensen to study reentrance from a paramagnetic to a ferromagnetic phase [44]. A model with frozen exchanges may be studied on a finite dimensional hierarchical lattice, which could be a first step in addressing the phase diagram in Figure 4b in a finite dimension. This approach should probably rely on a numerical iteration of the renormalization equations similar to reference [44] while an analytic study was possible in the present work.

Finally, ordering by disorder seems to be a generic behavior of spin liquids [27,36,28,45,46]. We found in the present work an ordering by disorder resulting in a transition from a paramagnetic to a spin glass ordering in an Ising model. This may be viewed as a "minimal" ordering by disorder from a Z_2 -symmetric paramagnet to a spin

glass because the Ising order parameter has the lowest possible spin symmetry.

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